Journal of Approximation Theory **96**, 399–404 (1999)
Article ID jath.1997.3249, available online at http://www.idealibrary.com on IDEAL®

On Boundedness of Lagrange Interpolation in L_p, $p<1$

D. S. Lubinsky

Department of Mathematics, Witwatersrand University, Wits 2050, South Africa
E-mail: 036dsl@cosmos.wits.ac.za

Communicated by Paul Nevai

Received November 13, 1997; accepted in revised form January 9, 1998

We estimate the distribution function of a Lagrange interpolation polynomial and deduce mean boundedness in L_p, $p<1$. © 1999 Academic Press

1. THE RESULT

There is a vast literature on mean convergence of Lagrange interpolation, see [4–8] for recent references. In this note, we use distribution functions to investigate mean convergence. We believe the simplicity of the approach merits attention.

Recall that if $g\colon \mathbb{R}\to\mathbb{R}$, and m denotes Lebesgue measure, then the *distribution function* m_g of g is

$$m_g(\lambda) := m(\{x\colon |g(x)|>\lambda\}), \qquad \lambda\geqslant 0. \tag{1}$$

One of the uses of m_g is in the identity [1, p. 43]

$$\|g\|^p_{L_p(\mathbb{R})} = \int_0^\infty pt^{p-1} m_g(t)\,dt, \qquad 0<p<\infty. \tag{2}$$

Moreover, the weak L_1 norm of g may be defined by

$$\|g\|_{weak(L_1)} = \sup_{\lambda>0} \lambda m_g(\lambda). \tag{3}$$

If

$$\|g\|_{L_p(\mathbb{R})}<\infty,$$

then for $p<\infty$, it is easily seen that

$$m_g(\lambda)\leqslant \lambda^{-p}\,\|g\|^p_{L_p(\mathbb{R})}, \qquad \lambda>0, \tag{4}$$

0021-9045/99 $30.00

and if $p=\infty$,

$$m_g(\lambda)=0, \qquad \lambda>\|g\|_{L_\infty(\mathbb{R})}.$$

Our result is:

THEOREM 1. *Let $w, v\colon \mathbb{R}\to\mathbb{R}$ be measurable and let v have compact support. Let $n\geqslant 1$ and let π_n be a polynomial of degree n with n real simple zeros $\{t_{jn}\}_{j=1}^n$. Let*

$$\Omega_n := \sum_{j=1}^n \frac{1}{|\pi_n' w|\,(t_{jn})}. \tag{5}$$

(a) *Let $0<r<\infty$ and assume there exists $A>0$ such that*

$$m_{\pi_n v}(\lambda)\leqslant A\lambda^{-r}, \qquad \lambda>0. \tag{6}$$

Then if $L_n[f]$ denotes the Lagrange interpolation polynomial to f at the zeros $\{t_{jn}\}$ of π_n, we have

$$m_{L_n[f]v}(\lambda)\leqslant 2A^{1/(r+1)}(8\,\|fw\|_{L_\infty(\mathbb{R})}\Omega_n/\lambda)^{r/(r+1)}, \qquad \lambda>0. \tag{7}$$

(b) *Assume that*

$$m_{\pi_n v}(\lambda)=0, \qquad \lambda>A. \tag{8}$$

Then

$$m_{L_n[f]\,v}(\lambda)\leqslant A\,\|fw\|_{L_\infty(\mathbb{R})}\Omega_n/\lambda, \qquad \lambda>0. \tag{9}$$

COROLLARY 2. *Let w, v be as in Theorem 1 and assume that we are given π_n, $\{t_{jn}\}_{j=1}^n$ for each $n\geqslant 1$ and*

$$\Omega := \sup_{n\geqslant 1}\sum_{j=1}^n \frac{1}{|\pi_n' w|\,(t_{jn})}<\infty. \tag{10}$$

(a) *If $r<\infty$ and (6) holds for $n\geqslant 1$, then for $0<p<r/(1+r)$, we have for some C_1 independent of f, n*

$$\|L_n[f]\,v\|_{L_p(\mathbb{R})}\leqslant C_1\,\|fw\|_{L_\infty(\mathbb{R})}. \tag{11}$$

(b) *If (8) holds for $n\geqslant 1$, then we have (11) for $0<p<1$, as well as*

$$\|L_n[f]\,v\|_{weak(L_1)}\leqslant C_1\,\|fw\|_{L_\infty(\mathbb{R})}. \tag{12}$$

Remarks. (a) Note that (6) holds if

$$\|\pi_n v\|^r_{L_r(\mathbb{R})} \leqslant A, \qquad n \geqslant 1$$

and (8) holds if

$$\|\pi_n v\|_{L_\infty(\mathbb{R})} \leqslant A.$$

Of course (6) is a weak L_r condition.

(b) Under mild additional conditions on w and v that guarantee density of the polynomials in the relevant spaces, the projection property $L_n[P] = P$, $\deg(P) \leqslant n-1$, allows us to deduce mean convergence of $L_n[f]$ to f.

(c) Orthogonal polynomials $\{p_n(u, x)\}_{n=0}^{\infty}$ such as those for generalized Jacobi weights u [4] or the exponential weights u in [2] admit the bound

$$|p_n(u, x)|\, u^{1/2}(x) \leqslant C \left|1 - \frac{|x|}{a_n}\right|^{-1/4}, \qquad x \in [-1, 1]$$

for a C independent of n and a suitable choice of a_n. Thus these polynomials admit the bound (6) with $r = 4$. Moreover, if $\{t_{jn}\}$ are the zeros of p_n, then a great deal is known about $p_n'(t_{jn})$, and in particular (10) holds with an appropriate choice of w. More generally, for extended Lagrange interpolation, involving interpolation at the zeros of $S_n p_n$, where S_n is a polynomial of fixed degree, it is easy to verify (10) under mild conditions on S_n.

(d) A result of Shi [7] implies that if (11) holds with C_1 independent of f and n, and if π_n is normalized by the condition

$$\|\pi_n v\|_{L_p(\mathbb{R})} = 1,$$

while the $\{t_{jn}\}$ are all contained in a bounded interval, then (10) holds. Thus in this case (10) is necessary for (11). However, our normalisation (6) or (8) of π_n involves a condition with $r > p$, so there is a gap.

(e) Of course (10) requires $w(t_{jn}) \neq 0\ \forall j, n$. We may weaken (10) to

$$\sup_{n \geqslant 1} \sum_{j:\, w(t_{jn}) \neq 0} \frac{1}{|\pi_n' w|\,(t_{jn})} < \infty$$

if we restrict f by the condition $w(t_{jn}) = 0 \Rightarrow f(t_{jn}) = 0$. In particular this allows us to consider w with compact support even when $\{t_{jn}\}_{j,n}$ is not contained in a bounded interval.

Our proofs rely on a lemma of Loomis [1, p. 129].

LEMMA 3. *Let $n \geqslant 1$ and $\{x_j\}_{j=1}^n, \{c_j\}_{j=1}^n \subset \mathbb{R}$. Then for $\lambda > 0$,*

$$m\left(\left\{x\colon \left|\sum_{j=1}^n \frac{c_j}{x-x_j}\right| > \lambda\right\}\right) \leqslant \frac{8}{\lambda} \sum_{j=1}^n |c_j|. \tag{13}$$

Proof. When all $c_j \geqslant 0$, we have equality in (13) with 8 replaced by 2 [1, p. 129]. The general case follows by writing

$$c_j = c_j^+ - c_j^-,$$

where $c_j^+ = \max\{0, c_j\}$, $c_j^- = -\min\{0, c_j\}$ and noting that

$$\left|\sum_{j=1}^n \frac{c_j}{x-x_j}\right| > \lambda \Rightarrow \left|\sum_{j=1}^n \frac{c_j^+}{x-x_j}\right| > \frac{\lambda}{2} \quad \text{or} \quad \left|\sum_{j=1}^n \frac{c_j^-}{x-x_j}\right| > \frac{\lambda}{2} \quad \text{or both.} \quad \blacksquare$$

Proof of Theorem 1. (a) Assume that $r < \infty$ and let $a \in \mathbb{R}$, $\lambda > 0$. We may assume that

$$\|fw\|_{L_\infty(\mathbb{R})} = 1. \tag{14}$$

(The general case follows from the identity $m_{bg}(\lambda) = m_g(\lambda/b)$ for $b, \lambda > 0$.) Now

$$(L_n[f]\, v)(x) = (\pi_n v)(x) \sum_{j=1}^n \frac{(fw)(t_{jn})}{(\pi_n' w)(t_{jn})(x - t_{jn})}$$

so

$$|L_n[f]\, v|\,(x) > \lambda$$

implies

$$|\pi_n v|\,(x) > \lambda^a \tag{15}$$

or

$$\left|\sum_{j=1}^n \frac{(fw)(t_{jn})}{(\pi_n' w)(t_{jn})(x - t_{jn})}\right| > \lambda^{1-a} \tag{16}$$

or both. The set of x satisfying (15) has, by (6), measure at most $A\lambda^{-ar}$. The set of x satisfying (16) has by Loomis' Lemma, measure at most

$$\frac{8}{\lambda^{1-a}} \sum_{j=1}^n \left|\frac{fw}{\pi_n' w}\right| (t_{jn}) \leqslant 8\lambda^{a-1} \Omega_n.$$

Now, if $\lambda \neq 1$, we choose a so that

$$A\lambda^{-ar} = 8\lambda^{a-1}\Omega_n \Leftrightarrow a = \frac{1}{r+1}\left[1 - \frac{\log[8\Omega_n/A]}{\log\lambda}\right].$$

Then we obtain

$$m_{L_n[f]\,v}(\lambda) \leqslant 2A^{1/(r+1)}(8\Omega_n/\lambda)^{r/(r+1)},$$

that is, (7) holds. The case $\lambda = 1$ follows from continuity properties of Lebesgue measure.

(b) Here we have instead

$$|L_n[f]\,v|\,(x) > \lambda \Rightarrow \left|\sum_{j=1}^{n} \frac{(fw)(t_{jn})}{(\pi_n' w)(t_{jn})(x - t_{jn})}\right| > \frac{\lambda}{A}$$

and again (9) follows from Loomis' Lemma. ∎

Proof of Corollary 2. (a) We may assume (14). Now by hypothesis, there exists $b > 0$ such that v vanishes outside $[-b, b]$. Thus in addition to (7), we have the estimate

$$m_{L_n[f]\,v}(\lambda) \leqslant 2b, \qquad \lambda > 0.$$

Then from (2), if $0 < p < r/(r+1)$, we have

$$\|L_n[f]\,v\|^p_{L_p(\mathbb{R})} \leqslant p\left(\int_0^1 t^{p-1}(2b)\,dt + 2A^{1/(r+1)}(8\Omega)^{r/(r+1)} \times \int_1^\infty t^{p-1-r/(r+1)}\,dt\right) =: C_1 < \infty.$$

(b) Here trivial modifications of this last estimate allows us to treat $0 < p < 1$, while (9) gives

$$\|L_n[f]\,v\|_{weak(L_1)} = \sup_{\lambda > 0} \lambda m_{L_n[f]\,v}(\lambda) \leqslant C\Omega. \quad ∎$$

We make two final remarks: The proof of Theorem 1 also gives a weak converse Marcinkiewicz–Zygmund inequality. For a given f, define

$$\Omega_n(f) := \sum_{j=1}^{n} \frac{|fw|\,(t_{jn})}{|\pi_n' w|\,(t_{jn})}.$$

Then (7) holds with Ω_n replaced by $\Omega_n(f)$. Moreover, (7) can be reformulated in the following way: If P is a polynomial of degree $\leqslant n-1$ satisfying

$$|Pw|\,(t_{jn}) \leqslant 1, \qquad 1 \leqslant j \leqslant n,$$

then

$$m_{Pv}(\lambda) \leqslant 2A^{1/(r+1)}(8\Omega_n/\lambda)^{r/(r+1)}, \qquad \lambda > 0.$$

It would be useful to have more sophisticated estimates for m_{Pv}. For special weights w, v and points $\{t_{jn}\}$, converse quadrature sum inequalities imply these [4].

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